INJECTIVITY THEOREMS AND ALGEBRAIC CLOSURES OF GROUPS WITH COEFFICIENTS

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Abstract. Recently, Cochran and Harvey defined torsion-free derived series of groups and proved an injectivity theorem on the associated torsion-free quotients. We show that there is a universal construction which extends such an injectivity theorem to an isomorphism theorem. Our result relates injectivity theorems to a certain homology localization of groups. In order to give a concrete combinatorial description and existence proof of the necessary homology localization, we introduce a new version of algebraic closures of groups with coefficients by considering a certain type of equations.

1. Introduction

Let R be a subring of the rationals. A map $f\colon X \to Y$ between two spaces X and Y is called an *R-homology equivalence* if f induces isomorphisms on $H_*(-;R)$. Homology equivalences play an important role in the study of various problems in geometric topology, including homology cobordism of manifolds and concordance of embeddings, in particular knot and link concordance. In this regard, a central question is how the fundamental groups of homology equivalent spaces relate. As a preliminary observation, it can be easily seen that an R-homology equivalence induces a homomorphism on the fundamental groups which is 2-connected on R-homology; we say that a group homomorphism ϕ is *2-connected on R-homology* if ϕ induces an isomorphism on $H_1(-;R)$ and a surjection on $H_2(-;R)$. When $R = \mathbb{Z}$ (resp. $\mathbb{Q}$), ϕ is called *integrally (resp. rationally) 2-connected*. We also remark that in most applications it suffices to consider finite complexes (for example, compact manifolds) and so fundamental groups can be assumed to be finitely presented.

Probably the first landmark result on the relationship between homology equivalences and fundamental groups is an isomorphism theorem of Stallings, which says that an integrally 2-connected homomorphism $\pi \to G$ induces an isomorphism $\pi/\pi_q \to G/G_q$, where G_q denotes the q-th lower central subgroup of G [15]. This has several well-known topological applications: abelian invariants such as linking numbers can be viewed as an application of the simplest nontrivial case $G/G_2 = H_1(G)$ of Stallings' theorem. More generally, the invariance of G/G_q plays a key role in understanding Milnor's $\bar{\mu}$-invariants of links [11, 12] as shown in Casson's work [2]. Orr defined further homotopy invariants of links using Stallings' theorem [13, 14]. In some cases homology cobordism invariants can be obtained by

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combining Stallings' theorem and the index theorem in a similar way as Levine's work on Atiyah–Patodi–Singer signatures of links [10]. In [6], Friedl applied this method to reformulate and generalize a link concordance invariant obtained from certain nonabelian and irregular covers due to the author and Ko [3].

In 2004, Cochran and Harvey announced a remarkable discovery of an injectivity theorem relating the rational homology of a group G to a certain type of derived series $\{G^{(n)}_H\}$, which is called the *torsion-free derived series* [4]. (The series is due to Harvey [7]; for a precise definition of $G^{(n)}_H$, see [4, 7], or Section 3.) The main result of [4] can be stated as follows: if $\pi \to G$ is a rationally 2-connected homomorphism of a finitely generated group π into a finitely presented group G, then it induces an injection $\pi/\pi^{(n)}_H \to G/G^{(n)}_H$. It has interesting applications as illustrated in a recent result of Harvey; she obtained invariants of homology cobordism by combining the injectivity with rank invariants and $L^{(2)}$-signature invariants.

The advent of the injectivity theorem leads us to ask a natural question: can one extend the torsion-free quotient $G/G^{(n)}_H$ in such a way that an isomorphism is induced instead of an injection? More generally, when can such an injectivity theorem be extended to an isomorphism theorem? Regarding topological applications, we remark that the fundamental ideas of the applications of Stallings' theorem could be reused when one has an isomorphism theorem.

In this paper we study injectivity theorems and their extensions to isomorphism theorems in a general setting motivated from Cochran–Harvey's result. To formalize injectivity theorems, we introduce a notion of an *I-functor*. For an arbitrary coefficient ring $R \subset \mathbb{Q}$, we think of the collection Ω^R of homomorphisms of finitely generated groups into finitely presented groups which are 2-connected on R-homology. Roughly speaking, we define an *I-functor* $\mathcal{H}$ to be a functorial association $G \to \mathcal{H}(G)$ such that to each $\pi \to G$ in Ω^R, an injection $\mathcal{H}(\pi) \to \mathcal{H}(G)$ is associated. Of course the main example of an I-functor is $\mathcal{H}(G) = G/G^{(n)}_H$ where $R = \mathbb{Q}$. We also formalize an isomorphism theorem extending the injectivity as follows: a *container* of an I-functor $\mathcal{H}$ is defined to be another I-functor $\mathcal{F}$ such that $\mathcal{H}$ injects into $\mathcal{F}$, that is, $\mathcal{H}(G) \subset \mathcal{F}(G)$, and $\mathcal{F}$ associates an isomorphism to each morphism in Ω^R. (Because there is some technical sophistication, we postpone precise definitions to Section 2; here we just remark that everything is required to have certain functorial properties which are naturally expected.)

Then our question can be stated as whether there is a container of a given I-functor. Note that if one has a container, then it is easy to construct a larger container by extending it; for example, take the direct sum with a constant functor. So the most essential one is a minimal container. We prove the following result:

Theorem 1.1. *If an I-functor $\mathcal{H}$ commutes with limits, then there exists a container $\widehat{\mathcal{H}}$ of $\mathcal{H}$ which is universal (initial) in the following sense: if $\mathcal{F}$ is another container, then $\widehat{\mathcal{H}}$ injects into $\mathcal{F}$ in a unique way.*

In other words, $\widehat{\mathcal{H}}$ provides an isomorphism theorem which extends the injectivity of $\mathcal{H}$, and it is universal among such extensions. For a more precise statement, see

Section 2. We remark that from its universality it follows that $\widehat{\mathcal{H}}$ is a unique minimal container of $\mathcal{H}$.

As a corollary of Theorem 1.1, we show that the torsion-free derived quotient $G \to G/G_H^{(n)}$ has a universal container (see Corollary 3.1). We remark that this special case was partially addressed in [4]; they constructed a container of $G/G_H^{(n)}$ by using an iterated semidirect product of certain homology groups. We also show that the container in [4] fails to be universal in our sense (see Theorem 3.7). An interesting observation is that this is closely related to the use of the *Ore localization* of a group ring $\mathbb{Q}[G]$ of a poly-torsion-free-abelian group G in the construction of the container in [4]. In the Ore localization all nonzero elements are inverted, but it turns out that this is too excessive; in showing that our universal container is strictly smaller, it is illustrated that the unnecessarily inverted elements are ones in the kernel of the augmentation $\mathbb{Q}[G] \to \mathbb{Q}$. (See Section 3 for more details.) This gives us a motivation for expecting a similar but more natural theory using the *Cohn localization*, instead of the Ore localization.

On the other hand, in our results there is something beyond the existence of a universal container. It relates injectivity theorems to a certain localization functor of groups. In general, one can view a localization functor as a *universal* construction inverting a given collection Ω of morphisms in a category. (Our definition of a localization is given in Section 5.) So, if a localization with respect to Ω exists, it provides a natural isomorphism theorem for morphisms in Ω. That is, it associates an isomorphism to each morphism in Ω. In order to prove Theorem 1.1, we use a particular localization functor $G \to \widehat{G}$ with respect to the collection Ω^R considered above. In fact, in the proof of Theorem 1.1, we show that the universal container $\widehat{\mathcal{H}}$ of $\mathcal{H}$ is given by $\widehat{\mathcal{H}}(G) = \mathcal{H}(\widehat{G})$. This presents another point of view that the injective homomorphism induced by $\mathcal{H}$ can be regarded as a restriction of a natural isomorphism obtained from the localization; for any homomorphism in Ω^R, the induced isomorphism on the localization gives rise to an isomorphism on $\widehat{\mathcal{H}}(-)$, and the induced homomorphism on $\mathcal{H}(-)$ is its restriction.

We remark that several homology localizations of groups have been studied in the literature, including works of Bousfield [1], Vogel, Levine [9, 8], and Farjoun–Orr–Shelah [5]. In particular, in [9, 8] Levine introduced the notion of an algebraic closure of a group, which turns out to be equivalent to Vogel's localization with respect to integrally 2-connected homomorphisms from finitely generated groups into finitely presented groups which are normally surjective. The localization with respect to our Ω^R which is used to prove Theorem 1.1 is similar to that of Levine, but distinguished in two points: first the normal surjectivity condition is not required, and second, an arbitrary subring R of $\mathbb{Q}$ is used as homology coefficients. Although it may be regarded as folklore that there exists a localization with respect to Ω^R, we give an existence proof for concreteness since we could not find any published one in the literature. In addition our work provides a combinatorial description of the desired R-homology localization. For this purpose, we introduce a new version of algebraic closures of groups, modifying the idea of Farjoun, Orr, and Shelah [5]

and Levine [9, 8]. We think of a certain class of systems of equations over a group G which we call *R-nullhomologous*, and define an *(algebraic) R-closure* $\widehat{G}$ of G in terms of solubility of such systems of equations. We show the existence of an R-closure $\widehat{G}$ for any group G, and show that it is equivalent to the desired localization with respect to Ω^R.

The remaining part of this paper is organized as follows. In Section 2 we prove the existence of a universal container, assuming the existence of the R-closure of a group. In Section 3 we apply the results of Section 2 to the case of the torsion-free derived quotient, and show that our universal container is strictly smaller than a container constructed in [4]. In the remaining sections, we prove results on the R-closure which are used in previous sections. In Section 4 we introduce the notion of R-nullhomologous systems of equations, and in Section 5 it is related to the localization of groups with respect to Ω^R. In Section 6 we show the existence of the R-closure and some properties of the R-closure of a finitely presented group.

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2. I-functors and containers

We start with a formalization of the notion of a container of the torsion-free derived quotient $G/G^{(n)}_H$. Here we have a technical issue that the association $G \to G/G^{(n)}_H$ is not a functor of the category $\mathcal{G}$ of groups; not all group homomorphisms induce a morphism on the quotients, although the result of Cochran–Harvey guarantees that a rationally 2-connected homomorphism of a finitely generated group into a finitely presented group gives rise to an induced homomorphism.

This leads us to consider what follows. Recall that for any subring R of $\mathbb{Q}$, we denote by Ω^R the class of group homomorphisms $\alpha\colon \pi \to G$ such that π is finitely generated, G is finitely presented, and α is 2-connected on R-homology, that is, α induces an isomorphism on $H_1(-;R)$ and a surjection on $H_2(-;R)$. Denoting $\mathcal{H}(G) = G/G^{(n)}_H$, in [4] it was shown that $\mathcal{H}$ has the following properties for $R = \mathbb{Q}$:

(1) To each group G, a homomorphism $p_G\colon G \to \mathcal{H}(G)$ is associated.
(2) To each homomorphism $\alpha\colon \pi \to G$ in Ω^R, an injection $\mathcal{H}(\pi) \to \mathcal{H}(G)$ is associated.
(3) The above associations have naturality, that is, $\mathcal{H}(\beta \circ \alpha) = \mathcal{H}(\beta) \circ \mathcal{H}(\alpha)$, $\mathcal{H}(\alpha) \circ p_\pi = p_G \circ \alpha$ (see the diagrams below), and $\mathcal{H}(\mathrm{id}_G) = \mathrm{id}_{\mathcal{H}(G)}$ whenever

the involved homomorphisms exist.

$$\begin{tikzcd} \mathcal{H}(\pi) \arrow[rr, "\mathcal{H}(\beta\circ\alpha)"] \arrow[dr, "\mathcal{H}(\alpha)"'] & & \mathcal{H}(P) \\ & \mathcal{H}(G) \arrow[ur, "\mathcal{H}(\beta)"'] & \end{tikzcd} \qquad \begin{tikzcd} \pi \arrow[r, "\alpha"] \arrow[d, "p_\pi"'] & G \arrow[d, "p_G"] \\ \mathcal{H}(\pi) \arrow[r, "\mathcal{H}(\alpha)"'] & \mathcal{H}(G) \end{tikzcd}$$

While $\mathcal{H}$ is not a functor of the category $\mathcal{G}$ of groups, we can view $\mathcal{H}$ as a functor of a subcategory of finitely presented groups, which are of our main interest regarding topological applications: let $\mathcal{G}^R$ be the category whose objects are finitely presented groups and whose morphisms are homomorphisms between finitely presented groups which are 2-connected on R-homology. Then from the above properties it follows that $\mathcal{H}$ induces a functor $\mathcal{G}^R \to \mathcal{G}$. Also p induces a natural transformation from the obvious inclusion functor $\mathcal{G}^R \to \mathcal{G}$ to (the functor induced by) $\mathcal{H}$. In particular, homology equivalences between finite complexes give rise to morphisms in $\mathcal{G}^R$ and then one can apply $\mathcal{H}$ to obtain injections.

One more obvious property of $\mathcal{H}$ which might be easily overlooked is the following:

(4) To any isomorphism $\alpha\colon \pi \to G$, an isomorphism $\mathcal{H}(\alpha)\colon \mathcal{H}(\pi) \to \mathcal{H}(G)$ is associated.

We remark that (4) does not follow from (1)–(3) since (1)–(3) do not guarantee that $\mathcal{H}(\alpha)$ is defined for an isomorphism α in general.

Results of this section are not specific to the torsion-free quotients; we consider any association $\mathcal{H}$ with the properties above.

Definition 2.1. $(\mathcal{H}, p)$ is called an *I-functor with respect to R-coefficients* if the above (1)–(4) are satisfied.

Note that if R' is a subring of R, then an I-functor with respect to R-coefficients is automatically an I-functor with respect to R'-coefficients. When the coefficient ring R is clearly understood, we simply say that $\mathcal{H}$ is an I-functor.

I-functors form (objects of) a category; a morphism τ between two I-functors $\mathcal{H}$ and $\mathcal{H}'$ is defined to be a natural transformation $\tau\colon \mathcal{H} \to \mathcal{H}'$, where $\mathcal{H}$ and $\mathcal{H}'$ are viewed as functors $\mathcal{G}^R \to \mathcal{G}$ as an abuse of notation, such that the diagram

$$\begin{tikzcd} G \arrow[d, "p_G"'] \arrow[dr, "p'_G"] & \\ \mathcal{H}(G) \arrow[r, "\tau_G"'] & \mathcal{H}'(G) \end{tikzcd}$$

commute for each object G in $\mathcal{G}^R$ where p_G and p'_G are the natural transformations that the I-functors $\mathcal{H}$ and $\mathcal{H}'$ are endowed with, respectively. If each τ_G is injective, then we say that τ is *injective* and $\mathcal{H}'$ is an *extension* of $\mathcal{H}$.

Definition 2.2. For an I-functor $\mathcal{H}$, a morphism $\tau\colon \mathcal{H} \to \mathcal{F}$ into another I-functor $\mathcal{F}$ is called a *container* of $\mathcal{H}$ if τ is injective and $\mathcal{F}(\alpha)\colon \mathcal{F}(\pi) \to \mathcal{F}(G)$ is an isomorphism for any morphism $\alpha\colon \pi \to G$ in Ω^R.

Sometimes we say that $\mathcal{F}$ is a container of $\mathcal{H}$ when we do not have to specify $\tau\colon \mathcal{H} \to \mathcal{F}$ explicitly.

As mentioned in the introduction, we are interested in a universal (initial) container of a given I-functor $\mathcal{H}$. To give its definition, we consider the category of containers and injective morphisms; objects are containers $\mathcal{F}$ of $\mathcal{H}$, and morphisms from $\mathcal{F}$ to $\mathcal{F}'$ are injective morphisms $\mathcal{F} \to \mathcal{F}'$ between the two I-functors $\mathcal{F}$ and $\mathcal{F}'$ which makes the diagram

$$\begin{tikzcd} \mathcal{H} \arrow[d] \arrow[dr] & \\ \mathcal{F} \arrow[r] & \mathcal{F}' \end{tikzcd}$$

commute.

Definition 2.3. A universal (initial) object $\mathcal{F}$ in the category of containers of $\mathcal{H}$ is called a *universal* container of $\mathcal{H}$, that is, for any container $\mathcal{F}'$ of $\mathcal{H}$, there is a unique morphism from $\mathcal{F}$ to $\mathcal{F}'$.

Obviously a universal container is unique if it exists. Also, a universal container is automatically minimal, in the sense that it is not a proper extension of another container. So if a universal container exists, it is a unique minimal container.

For our existence result of a universal container, we need to formulate a relationship of an I-functor $\mathcal{H}$ and limits. In this paper it suffices to consider the direct limit of a sequence

$$G_0 \longrightarrow G_1 \longrightarrow G_2 \longrightarrow \cdots.$$

of group homomorphisms in $\mathcal{G}^R$. Usually, if $\mathcal{H}$ were an ordinary functor $\mathcal{G} \to \mathcal{G}$, we would say that $\mathcal{H}$ commutes with limits when $\varinjlim \mathcal{H}(G_k) \cong \mathcal{H}(\varinjlim G_k)$; more precisely, the isomorphism is explicitly specified in this case. Namely, $G_k \to \varinjlim G_k$ induces $\mathcal{H}(G_k) \to \mathcal{H}(\varinjlim G_k)$, and then

$$\varinjlim \mathcal{H}(G_k) \longrightarrow \mathcal{H}(\varinjlim G_k)$$

is induced. If it is an isomorphism, then we say that $\mathcal{H}$ commutes with limits. However, in our case, because $\mathcal{H}$ is just an I-functor, the homomorphism $G_k \to \varinjlim G_k$ does not necessarily induce $\mathcal{H}(G_k) \to \mathcal{H}(\varinjlim G_k)$ in general. So we need to adopt the existence of this induced homomorphism as a part of a definition:

Definition 2.4. An I-functor $\mathcal{H}$ is said to *commute with limits* if for any sequence

$$G_0 \longrightarrow G_1 \longrightarrow G_2 \longrightarrow \cdots$$

of morphisms $G_k \to G_{k+1}$ in $\mathcal{G}^R$, $\mathcal{H}$ associates to $G_k \to \varinjlim G_k$ a homomorphism $\mathcal{H}(G_k) \to \mathcal{H}(\varinjlim G_k)$, and its limit

$$\varinjlim \mathcal{H}(G_k) \longrightarrow \mathcal{H}(\varinjlim G_k)$$

is an isomorphism.

We note that even though each $G_k \to G_{k+1}$ is in $\mathcal{G}^R$, $\varinjlim G_k$ is not necessarily (an object) in $\mathcal{G}^R$.

Now we can precisely state the main result of this section.

Theorem 2.5. *Suppose that R is a subring of $\mathbb{Q}$ and $\mathcal{H}$ is an I-functor with respect to R-coefficients which commutes with limits. Then there exists a universal container $\tau\colon \mathcal{H} \to \widehat{\mathcal{H}}$ of $\mathcal{H}$, that is, for any container $\sigma\colon \mathcal{H} \to \mathcal{F}$, there is a unique injective morphism $\widehat{\sigma}\colon \widehat{\mathcal{H}} \to \mathcal{F}$ such that the diagram*

$$\begin{tikzcd}
& G \arrow[d] \arrow[ddl] \arrow[ddr] & \\
& \widehat{\mathcal{H}}(G) \arrow[dr, dotted, "\widehat{\sigma}_G"] & \\
\mathcal{H}(G) \arrow[ur, "\tau_G"] \arrow[rr, "\sigma_G"'] & & \mathcal{F}(G)
\end{tikzcd}$$

commutes.

To prove Theorem 2.5, we use a homology localization functor $E\colon \mathcal{G} \to \mathcal{G}$ with respect to R-coefficients. At this moment we just need the following properties of E, which are analogues of Levine's results on algebraic closures of groups [9, 8]; a construction of our E and proofs of the necessary properties are postponed to later sections.

Theorem 2.6. *For any subring R of $\mathbb{Q}$, there is a pair (E, i) of a functor $E\colon \mathcal{G} \to \mathcal{G}$ and a natural transformation $i\colon \mathrm{id}_{\mathcal{G}} \to E$ which has the following properties:*

1. *For any $\alpha\colon \pi \to G$ in Ω^R, the induced homomorphism $E(\alpha)\colon E(\pi) \to E(G)$ is an isomorphism.*
2. *For any object G in $\mathcal{G}^R$, there is a sequence*
$$G = G_0 \longrightarrow G_1 \longrightarrow \cdots \longrightarrow G_k \longrightarrow \cdots$$
of morphisms $G_k \to G_{k+1}$ in $\mathcal{G}^R$ such that $E(G) = \varinjlim G_k$ and $i_G\colon G \to E(G)$ is the limit homomorphism.

We denote $E(G)$ by $\widehat{G}$. For any I-functor $\mathcal{H}$ which commutes with limits, we will prove that the composition of E and $\mathcal{H}$ is a universal container of $\mathcal{H}$.

Proof of Theorem 2.5. We define $\widehat{\mathcal{H}}(G) = \mathcal{H}(\widehat{G})$ and $\widehat{p}_G\colon G \to \widehat{\mathcal{H}}(G)$ to be the composition
$$G \xrightarrow{i_G} \widehat{G} \xrightarrow{p_{\widehat{G}}} \mathcal{H}(\widehat{G}) = \widehat{\mathcal{H}}(G).$$
In other words, $\widehat{\mathcal{H}} = \mathcal{H} \circ E$ and $\widehat{p} = p \circ i$. We will show that $(\widehat{\mathcal{H}}, \widehat{p})$ is an I-functor. For any $\alpha\colon \pi \to G$ which is in Ω^R, $\widehat{\alpha}\colon \widehat{\pi} \to \widehat{G}$ is an isomorphism. Applying $\mathcal{H}$,

we obtain an induced isomorphism $\mathcal{H}(\widehat{\pi}) \to \mathcal{H}(\widehat{G})$. We define $\widehat{\mathcal{H}}(\alpha)$ to be this isomorphism. Viewing $(\widehat{H}, \widehat{p})$ as $(\mathcal{H} \circ E, p \circ i)$, the required naturality of $(\widehat{H}, \widehat{p})$ follows from that of $(\mathcal{H}, p)$ and (E, i).

For a finitely presented group G, $(\widehat{\mathcal{H}}, \widehat{p})$ can be interpreted as follows. Choose a sequence

$$G = G_0 \longrightarrow G_1 \longrightarrow G_2 \longrightarrow \cdots$$

of morphisms in $\mathcal{G}^R$ such that $i_G \colon G \to \widehat{G}$ is the limit homomorphism $G \to \varinjlim G_k \cong \widehat{G}$. By the naturality of p,

$$\begin{tikzcd} G \arrow[r] \arrow[d, "p_G"'] & G_k \arrow[r] \arrow[d, "p_{G_k}"] & \varinjlim G_k \arrow[d, "p_{\varinjlim G_k}"] \\ \mathcal{H}(G) \arrow[r] & \mathcal{H}(G_k) \arrow[r] & \mathcal{H}(\varinjlim G_k) \end{tikzcd}$$

commutes. Taking the limit, we have a commutative diagram

$$\begin{tikzcd} G \arrow[r, "i_G"] \arrow[d, "p_G"'] & \varinjlim G_k \arrow[r, equal] \arrow[d, "\varinjlim p_{G_k}"] & \varinjlim G_k = \widehat{G} \arrow[d, "p_{\varinjlim G_k}"] \\ \mathcal{H}(G) \arrow[r, "\text{limit map}"'] & \varinjlim \mathcal{H}(G_k) \arrow[r, "\cong"'] & \mathcal{H}(\varinjlim G_k) = \mathcal{H}(\widehat{G}) \end{tikzcd}$$

That is, $\widehat{\mathcal{H}}(G) = \varinjlim \mathcal{H}(G_k)$ and $\mathcal{H}$ associates to i_G the limit homomorphism

$$\mathcal{H}(i_G) \colon \mathcal{H}(G) \longrightarrow \widehat{\mathcal{H}}(G) = \varinjlim \mathcal{H}(G_k).$$

Also, $\widehat{p}_G \colon G \to \widehat{\mathcal{H}}(G)$ is the composition

$$G \xrightarrow{p_G} \mathcal{H}(G) \xrightarrow{\mathcal{H}(i_G)} \varinjlim \mathcal{H}(G_k).$$

Now we construct an injective morphism $\tau \colon (\mathcal{H}, p) \to (\widehat{\mathcal{H}}, \widehat{p})$ between the I-functors $(\mathcal{H}, p)$ and $(\widehat{\mathcal{H}}, \widehat{p})$ as follows. For a finitely presented group G, there exists $\mathcal{H}(i_G) \colon \mathcal{H}(G) \to \mathcal{H}(\widehat{G})$ as discussed above. We define $\tau_G \colon \mathcal{H}(G) \to \widehat{\mathcal{H}}(G)$ to be $\mathcal{H}(i_G)$, that is, $\tau = \mathcal{H} \circ i$. Viewing τ as a transformation between functors $\mathcal{H}, \mathcal{H}' \colon \mathcal{G}^R \to \mathcal{G}$, the naturality of τ follows from that of $\mathcal{H}$ and i. Furthermore,

$$\begin{tikzcd} G \arrow[d, "p_G"'] \arrow[dr, "\widehat{p}_G"] & \\ \mathcal{H}(G) \arrow[r, "\tau_G"'] & \widehat{\mathcal{H}}(G) \end{tikzcd}$$

commutes since $\widehat{p}_G = \mathcal{H}(i_G) \circ p_G$. This shows that τ is a morphism $(\mathcal{H}, p) \to (\widehat{\mathcal{H}}, \widehat{p})$. To show the injectivity, we consider a sequence $G = G_0 \to G_1 \to \cdots$ with limit $\widehat{G}$ as above. Since $\mathcal{H}$ is an I-functor and $G \to G_k$ is in Ω^R, $\mathcal{H}(G) \to \mathcal{H}(G_k)$ is injective. Since $\tau_G = \mathcal{H}(i_G)$ is the limit of $\mathcal{H}(G) \to \mathcal{H}(G_k)$, τ_G is injective too.

We will show that $\tau\colon (\mathcal{H}, p) \to (\widehat{\mathcal{H}}, \widehat{p})$ has the universal property. Suppose that $\sigma\colon (\mathcal{H}, p) \to (\mathcal{F}, q)$ is a container. We define a morphism $\widehat{\sigma}\colon (\widehat{\mathcal{H}}, \widehat{p}) \to (\mathcal{F}, q)$ as follows: for a finitely presented group G, choose $G = G_0 \to G_1 \to \cdots$ with limit $\widehat{G}$ as above. Taking the limit of

$$\begin{tikzcd}
 & G \arrow[r] \arrow[d, "p_G"] \arrow[ddl, "q_G"'] & G_k \arrow[d, "p_{G_k}"'] \arrow[ddr, "q_{G_k}"] & \\
 & \mathcal{H}(G) \arrow[r, hook] \arrow[dl, hook, "\sigma_G"'] & \mathcal{H}(G_k) \arrow[dr, hook, "\sigma_{G_k}"] & \\
\mathcal{F}(G) \arrow[rrr, "\cong"'] & & & \mathcal{F}(G_k)
\end{tikzcd}$$

we obtain a commutative diagram

$$(*)_{\{G_k\}} \qquad \begin{tikzcd}
 & G \arrow[r] \arrow[d, "p_G"] \arrow[ddl, "q_G"'] & \varinjlim G_k \arrow[d, "\varinjlim p_{G_k}"'] \arrow[ddr, "\varinjlim q_{G_k}"] & \\
 & \mathcal{H}(G) \arrow[r, hook] \arrow[dl, hook, "\sigma_G"'] & \varinjlim \mathcal{H}(G_k) \arrow[dr, hook, "\varinjlim \sigma_{G_k}"] & \\
\mathcal{F}(G) \arrow[rrr, "\cong"'] & & & \varinjlim \mathcal{F}(G_k)
\end{tikzcd}$$

We define $\widehat{\sigma}_G$ to be $\varinjlim \sigma_{G_k}$, that is,

$$\widehat{\sigma}_G\colon \widehat{\mathcal{H}}(G) = \mathcal{H}(\widehat{G}) = \varinjlim \mathcal{H}(G_k) \xrightarrow{\varinjlim \sigma_{G_k}} \varinjlim \mathcal{F}(G_k) = \mathcal{F}(G).$$

Since each σ_{G_k} is injective, so is $\widehat{\sigma}_G$.

At the moment, our $\widehat{\sigma}_G$ depends on the choice of $\{G_k\}$. Before showing that it is well-defined, we prove the naturality of $\widehat{\sigma}$. Suppose $\alpha\colon \pi \to G$ is in $\mathcal{G}^R$ and $\pi = \pi_0 \to \pi_1 \to \cdots$ and $G = G_0 \to G_1 \to \cdots$ are sequences giving $\widehat{\pi}$ and $\widehat{G}$ as above. First we consider a special case that $\{\pi_k\}$ and $\{G_k\}$ behave nicely under α, that is, we suppose that there are homomorphisms $\pi_k \to G_k$ which fit into the

following commutative diagram:

$$\begin{array}{ccccccc}
\pi = \pi_0 & \longrightarrow & \pi_1 & \longrightarrow & \pi_2 & \longrightarrow & \cdots \\
\downarrow{\scriptstyle\alpha} & & \downarrow & & \downarrow & & \\
G = G_0 & \longrightarrow & G_1 & \longrightarrow & G_2 & \longrightarrow & \cdots
\end{array}$$

Then α, together with $\pi_k \to G_k$, induces a "morphism" between the diagrams $(*)_{\{\pi_k\}}$ and $(*)_{\{G_k\}}$. In particular, the morphisms $\widehat{\mathcal{H}}(\alpha)$ and $\mathcal{F}(\alpha)$ give us the commutative diagram below, which says that $\widehat{\sigma}$ is natural in this special case:

$$\begin{array}{ccc}
\widehat{\mathcal{H}}(\pi) & \xrightarrow{\widehat{\mathcal{H}}(\alpha)} & \widehat{\mathcal{H}}(G) \\
{\scriptstyle \varinjlim \sigma_{\pi_k} = \widehat{\sigma}_\pi}\downarrow & & \downarrow{\scriptstyle \widehat{\sigma}_G = \varinjlim \sigma_{G_k}} \\
\mathcal{F}(\pi) & \xrightarrow[\mathcal{F}(\alpha)]{} & \mathcal{F}(G)
\end{array}$$

To reduce the general case to the special case above, we appeal to the following result, which is a horseshoe-type lemma:

Lemma 2.7. *Suppose that $\pi = \pi_0 \to \pi_1 \to \cdots$ and $G = G_0 \to G_1 \to \cdots$ are sequences of morphisms in $\mathcal{G}^R$ such that $\widehat{\pi} = \varinjlim \pi_k$ and $\widehat{G} = \varinjlim G_k$, and $\alpha\colon \pi \to G$ is in $\mathcal{G}^R$. Then there exists a commutative diagram*

$$\begin{array}{ccccccccc}
\pi = \pi_0 & \longrightarrow & \pi_1 & \longrightarrow \cdots \longrightarrow & \pi_k & \longrightarrow & \cdots & \rightarrow & \varinjlim \pi_k = \widehat{\pi} \\
\downarrow{\scriptstyle\alpha} & & \downarrow & & \downarrow & & & & \downarrow \\
G = P_0 & \longrightarrow & P_1 & \longrightarrow \cdots \longrightarrow & P_k & \longrightarrow & \cdots & \rightarrow & \varinjlim P_k \\
\| & & \uparrow & & \uparrow & & & & \uparrow \\
G = G_0 & \longrightarrow & G_1 & \longrightarrow \cdots \longrightarrow & G_k & \longrightarrow & \cdots & \rightarrow & \varinjlim G_k = \widehat{G}
\end{array}$$

where $\pi_k \to P_k$, $G_k \to P_k$, and $P_k \to P_{k+1}$ are in $\mathcal{G}^R$,

$$\widehat{G} = \varinjlim G_k \longrightarrow \varinjlim P_k$$

is an isomorphism, and the limit homomorphism

$$G = P_0 \longrightarrow \varinjlim P_k \cong \widehat{G}$$

is equal to $i_G\colon G \to \widehat{G}$.

Applying the above special case to $(\{\pi_k\}, \{P_k\})$ and $(\{P_k\}, \{G_k\})$, we obtain a commutative diagram

$$(**)\qquad \begin{tikzcd} \widehat{\mathcal{H}}(\pi) \arrow[r, "\widehat{\mathcal{H}}(\alpha)"] \arrow[d, "\varinjlim \sigma_{\pi_k}"'] & \widehat{\mathcal{H}}(G) \arrow[r, "\widehat{\mathcal{H}}(\mathrm{id})=\mathrm{id}"] \arrow[d, "\varinjlim \sigma_{P_k}"] & \widehat{\mathcal{H}}(G) \arrow[d, "\varinjlim \sigma_{G_k}"] \\ \mathcal{F}(\pi) \arrow[r, "\mathcal{F}(\alpha)"'] & \mathcal{F}(G) \arrow[r, "\mathcal{F}(\mathrm{id})=\mathrm{id}"'] & \mathcal{F}(G) \end{tikzcd}$$

This shows that $\widehat{\sigma}$ behaves naturally for $\pi \to G$ even when $\widehat{\sigma}_\pi$ and $\widehat{\sigma}_G$ are defined using arbitrarily chosen $\{\pi_k\}$ and $\{G_k\}$.

We can also use the same argument to show the well-definedness of $\widehat{\sigma}_G$, that is, $\widehat{\sigma}_G$ is independent of the choice of $\{G_k\}$. For this, we apply Lemma 2.7 to a special case that $\pi = G$ and $\alpha\colon \pi \to G$ is the identity. Then for any $\{\pi_k\}$ and $\{G_k\}$ with limit $\widehat{G}$, there is $\{P_k\}$ which gives the diagram $(**)$. Since $\widehat{\mathcal{H}}(\alpha) = \mathrm{id}$ in this case, it follows that the homomorphisms

$$\varinjlim \sigma_{\pi_k},\ \varinjlim \sigma_{P_k},\ \varinjlim \sigma_{G_k} : \widehat{\mathcal{H}}(G) \longrightarrow \mathcal{F}(G)$$

are all equal.

From the diagram $(*)_{\{G_k\}}$, we obtain a commutative diagram

$$\begin{tikzcd} & G \arrow[d, "p_G"] \arrow[ddl, "\widehat{p}_G"'] \arrow[ddr, "q_G"] & \\ & \mathcal{H}(G) \arrow[dl, "\tau_G"] \arrow[dr, "\sigma_G"'] & \\ \widehat{\mathcal{H}}(G) \arrow[rr, "\widehat{\sigma}_G"'] & & \mathcal{F}(G) \end{tikzcd} .$$

From this it follows that $\widehat{\sigma}$ can be viewed as a morphism between containers.

Finally we show the uniqueness of $\widehat{\sigma}$. Suppose that $\widehat{\sigma}'\colon (\widehat{\mathcal{H}}, \widehat{p}) \to (\mathcal{F}, q)$ is another morphism between the containers $(\widehat{\mathcal{H}}, \widehat{p})$ and $(\mathcal{F}, q)$. For a sequence $G = G_0 \to$

$G_1 \to \cdots$ giving $\widehat{G}$, we have the following commutative diagram:

$$\begin{tikzcd}
 & \mathcal{H}(G) \arrow[dl, "\tau_G"'] \arrow[d, "\sigma_G"] \arrow[dr] & \\
\widehat{\mathcal{H}}(G) \arrow[r, "\widehat{\sigma}'_G"] \arrow[dr, "\cong"'] & \mathcal{F}(G) \arrow[dr, "\cong"] & \mathcal{H}(G_k) \arrow[dl, "\tau_{G_k}"'] \arrow[d, "\sigma_{G_k}"] \\
 & \varinjlim \mathcal{H}(G_k) = \widehat{\mathcal{H}}(G_k) \arrow[r, "\widehat{\sigma}'_{G_k}"'] & \mathcal{F}(G_k)
\end{tikzcd}$$

Since $\widehat{\sigma}'_{G_k}\tau_{G_k} = \sigma_{G_k} = \tau_{G_k}\widehat{\sigma}_{G_k}$, $\widehat{\sigma}'_G$ and $\widehat{\sigma}_G\colon \widehat{\mathcal{H}}(G) \to \mathcal{F}(G)$ coincide on the image of $\tau_{G_k}\colon \mathcal{H}(G_k) \to \widehat{\mathcal{H}}(G_k) = \widehat{\mathcal{H}}(G)$. From this it follows that $\widehat{\sigma}'_G = \widehat{\sigma}_G$ since $\widehat{\mathcal{H}}(G) = \varinjlim \mathcal{H}(G_k)$. $\square$

3. Containers of torsion-free derived series

In this section we focus on a special case of an I-functor, namely the torsion-free derived quotient $G \to G/G^{(n)}_H$. We begin by recalling the definition of $G^{(n)}_H$ in [4]. For a group G, $G^{(0)}_H$ is defined to be G itself. Suppose $G^{(n)}_H$ has been defined to be a normal subgroup of G such that that $G/G^{(n)}_H$ is a poly-torsion-free-abelian (PTFA) group. Since the integral group ring of a PTFA group is an Ore domain, there exists the skew field $\mathcal{K}[G/G^{(n)}_H]$ of (right) quotients of $\mathbb{Z}[G/G^{(n)}_H]$, that is, $\mathcal{K}[G/G^{(n)}_H] = \mathbb{Z}[G/G^{(n)}_H](\mathbb{Z}[G/G^{(n)}_H] - \{0\})^{-1}$. Note that $\mathcal{K}[G/G^{(n)}_H]$ is $\mathbb{Z}[G/G^{(n)}_H]$-flat. $G^{(n+1)}_H$ is defined to be the kernel of the following composition:

$$\begin{aligned} G^{(n)}_H &\longrightarrow G^{(n)}_H/[G^{(n)}_H, G^{(n)}_H] = H_1(G; \mathbb{Z}[G/G^{(n)}_H]) \\ &\longrightarrow H_1(G; \mathbb{Z}[G/G^{(n)}_H]) \underset{\mathbb{Z}[G/G^{(n)}_H]}{\otimes} \mathcal{K}[G/G^{(n)}_H] = H_1(G; \mathcal{K}[G/G^{(n)}_H]) \end{aligned}$$

Since $G/G^{(n+1)}_H$ is an extension of $G^{(n)}_H/G^{(n+1)}_H$ by $G/G^{(n)}_H$ and $G^{(n)}_H/G^{(n+1)}_H$ is a subgroup of $H_1(G; \mathcal{K}[G/G^{(n)}_H])$ which is a torsion-free abelian group, $G/G^{(n+1)}_H$ is PTFA so that one can continue this process. For $n = \omega$, the first infinite ordinal, $G^{(n)}_H$ is defined to be the intersection of $G^{(k)}_H$ where k runs over all integers. For further details see [4].

We denote $\mathcal{H}_n(G) = G/G^{(n)}_H$. It was shown in [4] that $\mathcal{H}_n$, equipped with the projection $G \to \mathcal{H}_n(G)$, is an I-functor with respect to $\mathbb{Q}$-coefficients. (In this section the coefficient ring R is always $\mathbb{Q}$.) From Theorem 2.5, it follows that $\mathcal{H}_n$ has a universal container:

Corollary 3.1. *There exists a universal container of $\mathcal{H}_n$ for any $n \leq \omega$.*

Proof. By Theorem 2.5, it suffices to show that $\mathcal{H}_n$ commutes with limits. Suppose that

$$G = G_0 \longrightarrow G_1 \longrightarrow G_2 \longrightarrow \cdots$$

is a sequence of morphisms in $\mathcal{G}^R$. We use the following two properties of $\mathcal{H}_n$: first, $G \to \mathcal{H}_n(G)$ is obviously surjective for any G, and second, we need Lemma 5.3 in [4]: the natural map $G_k \to \varinjlim G_k$ gives rise to an injection $\mathcal{H}_n(G_k) \to \mathcal{H}_n(\varinjlim G_k)$ for any k and any $n \leq \omega$.

Taking the limit of the injections $\mathcal{H}_n(G_k) \to \mathcal{H}_n(\varinjlim G_k)$, we obtain an injection

$$\varinjlim \mathcal{H}_n(G_k) \longrightarrow \mathcal{H}_n(\varinjlim G_k).$$

Since every x in $\mathcal{H}_n(\varinjlim G_k)$ is represented by an element in $\varinjlim G_k$, x is in the image of some G_k, and so in the image of some $\mathcal{H}_n(G_k)$. From this the surjectivity follows. □

Recall that the universal container $\widehat{\mathcal{H}}_n$ of $\mathcal{H}_n$ is given by $\widehat{\mathcal{H}}_n(G) = \mathcal{H}_n(\widehat{G})$, where $\widehat{G}$ denotes our homology localization with respect to $\mathbb{Q}$-coefficients given in Theorem 2.6.

Remark 3.2. For a $(4k-1)$-manifold M with fundamental group π, Harvey considered the $L^{(2)}$-signature of M associated to $\pi \to \mathcal{H}_n(\pi)$ as a homology cobordism invariant. Since $\mathcal{H}_n(\pi) \to \widehat{\mathcal{H}}_n(\pi) = \widehat{\pi}/\widehat{\pi}_H^{(n)}$ is injective by Corollary 3.1, from the induction property of the $L^{(2)}$-signature it follows that Harvey's invariant coincides with the $L^{(2)}$-signature associated to a characteristic quotient of the localization of π, namely $\pi \to \widehat{\pi} \to \widehat{\pi}/\widehat{\pi}_H^{(n)}$.

On the other hand, Cochran and Harvey defined a container $\mathcal{F}_n(G) = \widetilde{G}_n$ of $\mathcal{H}_n$ as follows. Let $\widetilde{G}_0 = \{e\}$, a trivial group. Suppose $\widetilde{G}_n$ has been defined as a PTFA group. Then $\widetilde{G}_{n+1}$ is defined to be $\widetilde{G}_{n+1} = H_1(G; \mathcal{K}\widetilde{G}_n) \rtimes \widetilde{G}_n$, where the semidirect product is formed by viewing $H_1(G; \mathcal{K}\widetilde{G}_n)$ as a $\mathbb{Z}[\widetilde{G}_n]$-module. Also, injections $\sigma_{n,G} \colon \mathcal{H}_n(G) = G/G_H^{(n)} \to \widetilde{G}_n = \mathcal{F}_n(G)$ are defined as follows. Initially $\sigma_{0,G}$ is the trivial homomorphism. Suppose $\sigma_{n,G}$ has been defined. Consider the composition

$$\phi_{n,G} \colon G_H^{(n)}/G_H^{(n+1)} \xrightarrow{\Phi_{n,G}} H_1(G; \mathcal{K}\mathcal{H}_n(G)) \xrightarrow{f} H_1(G; \mathcal{K}\mathcal{F}_n(G))$$

where $\Phi_{n,G}$ is the injection induced by the homomorphism used above to define $G_H^{(n+1)}$, and f is induced by $\sigma_{n,G} \colon \mathcal{H}_n(G) \to \mathcal{F}_n(G)$. In [4] the followings were shown: there is a derivation $G \to H_1(G; \mathcal{K}\mathcal{F}_n(G))$ which induces $\phi_{n,G}$. This derivation, together with $G \to \mathcal{H}_n(G) \to \mathcal{F}_n(G)$, gives rise to a homomorphism $G \to H_1(G; \mathcal{K}\mathcal{F}_n(G)) \rtimes \mathcal{F}_n(G) = \mathcal{F}_{n+1}(G)$ with kernel $G_H^{(n+1)}$. So it induces an injection $\sigma_{n+1,G} \colon \mathcal{H}_{n+1}(G) \to \mathcal{F}_{n+1}(G)$. For $n = \omega$, $\widetilde{G}_\omega$ (which is denoted by $\widetilde{G}$ and called the solvable completion of G in [4]) is defined to be $\widetilde{G}_\omega = \varprojlim \widetilde{G}_k$ where $k < \omega$. For any $n \leq \omega$, $\sigma_n \colon \mathcal{H}_n \to \mathcal{F}_n$ is a container of $\mathcal{H}_n$. For further details see [4].

We note that from the definitions it follows that there is a commutative diagram

$$(***)\qquad \begin{tikzcd} 1 \arrow[r] & G_H^{(n)}/G_H^{(n+1)} \arrow[r] \arrow[d, "\phi_{n,G}"] & \mathcal{H}_{n+1}(G) \arrow[r] \arrow[d, "\sigma_{n+1,G}"] & \mathcal{H}_n(G) \arrow[r] \arrow[d, "\sigma_{n,G}"] & 1 \\ 1 \arrow[r] & H_1(G;\mathcal{K}\mathcal{F}_n(G)) \arrow[r] & \mathcal{F}_{n+1}(G) \arrow[r] & \mathcal{F}_n(G) \arrow[r] & 1 \end{tikzcd}$$

with exact rows.

In the remaining part of this section, we compare the container $\mathcal{F}_n$ with the universal container $\widehat{\mathcal{H}}_n$ of $\mathcal{H}_n$. More precisely, by Corollary 3.1, there is an injective morphism $\widehat{\sigma}_{n,G}$ of our universal container $\widehat{\mathcal{H}}_n(G) = \widehat{G}/\widehat{G}_H^{(n)}$ to $\mathcal{F}_n(G) = \widetilde{G}_n$. Then our question is whether $\widehat{\sigma}_{n,G}$ is an isomorphism. The following proposition says that this is closely related to the structure of certain homology modules of $\widehat{G}$ in an inductive manner.

Proposition 3.3. *For a finitely presented group G, $\widehat{\sigma}_{n+1,G}$ is an isomorphism if and only if $\widehat{\sigma}_{n,G}$ is an isomorphism and the canonical homomorphism*

$$H_1(\widehat{G};\mathbb{Z}\mathcal{H}_n(\widehat{G})) \longrightarrow H_1(\widehat{G};\mathcal{K}\mathcal{H}_n(\widehat{G}))$$

is surjective.

Proof. Before proving the proposition, we assert that if $\widehat{\sigma}_{n,G}\colon \mathcal{H}_n(\widehat{G}) \to \mathcal{F}_n(G)$ is an isomorphism then there is a commutative diagram

$$\begin{tikzcd} 1 \arrow[r] & \widehat{G}_H^{(n)}/\widehat{G}_H^{(n+1)} \arrow[r] \arrow[d, "\Phi_{n,\widehat{G}}"] & \mathcal{H}_{n+1}(\widehat{G}) \arrow[r] \arrow[d, "\widehat{\sigma}_{n+1,G}"] & \mathcal{H}_n(\widehat{G}) \arrow[r] \arrow[d, "\widehat{\sigma}_{n,G}"] & 1 \\ 1 \arrow[r] & H_1(\widehat{G};\mathcal{K}\mathcal{H}_n(\widehat{G})) \arrow[r] & \mathcal{F}_{n+1}(G) \arrow[r] & \mathcal{F}_n(G) \arrow[r] & 1 \end{tikzcd}$$

with exact rows.

To prove this, we recall that the morphism $\widehat{\sigma}_n$ can be described as follows. Choose a sequence $G = G_0 \to G_1 \to \cdots$ of morphisms in $\Omega^{\mathbb{Q}}$ whose limit is $\widehat{G}$. Then $\mathcal{H}_n(\widehat{G}) \cong \varinjlim \mathcal{H}_n(G_k)$ and $\widehat{\sigma}_{n,G}\colon \mathcal{H}_n(\widehat{G}) \to \mathcal{F}_n(G)$ is the limit of

$$\sigma_{n,G_k}\colon \mathcal{H}_n(G_k) \longrightarrow \mathcal{F}_n(G_k) \cong \mathcal{F}_n(G)$$

as $k \to \infty$.

By the hypothesis of the assertion, $\mathcal{H}_n(\widehat{G}) \cong \mathcal{F}_n(G) \cong \mathcal{F}_n(G_k)$ where the latter isomorphism is induced by $G \to G_k$. So from the diagram $(***)$ we obtain a

commutative diagram

$$\begin{tikzcd}
1 \arrow[r] & \dfrac{G_H^{(n)}}{G_H^{(n+1)}} \arrow[rr] \arrow[dr] \arrow[dd, "\phi_{n,G}"'] & & \mathcal{H}_{n+1}(G) \arrow[rr] \arrow[dr] \arrow[dd, "\sigma_{n+1,G}"] & & \mathcal{H}_n(G) \arrow[r] \arrow[dr] \arrow[dd, "\sigma_{n,G}"] & 1 & \\
& 1 \arrow[r] & \dfrac{(G_k)_H^{(n)}}{(G_k)_H^{(n+1)}} \arrow[rr] \arrow[dd, "\phi_{n,G_k}"] & & \mathcal{H}_{n+1}(G_k) \arrow[rr] \arrow[dd, "\sigma_{n+1,G_k}"] & & \mathcal{H}_n(G_k) \arrow[r] \arrow[dd, "\sigma_{n,G_k}"] & 1 \\
1 \arrow[r] & H_1(G;\mathcal{K}\mathcal{H}_n(\widehat{G})) \arrow[rr] \arrow[dr] & & \mathcal{F}_{n+1}(G) \arrow[rr] \arrow[dr, "\cong"'] & & \mathcal{F}_n(G) \arrow[r] \arrow[dr, "\cong"'] & 1 & \\
& 1 \arrow[r] & H_1(G_k;\mathcal{K}\mathcal{H}_n(\widehat{G})) \arrow[rr] & & \mathcal{F}_{n+1}(G_k) \arrow[rr] & & \mathcal{F}_n(G_k) \arrow[r] & 1
\end{tikzcd}$$

with exact rows. $H_1(G;\mathcal{K}\mathcal{H}_n(\widehat{G})) \to H_1(G_k;\mathcal{K}\mathcal{H}_n(\widehat{G}))$ is an isomorphism since $\mathcal{F}_{n+1}(G) \to \mathcal{F}_{n+1}(G_k)$ and $\mathcal{F}_n(G) \to \mathcal{F}_n(G_k)$ are isomorphisms. Since H_1 commutes with limits,

$$H_1(G;\mathcal{K}\mathcal{H}_n(\widehat{G})) = \varinjlim H_1(G_k;\mathcal{K}\mathcal{H}_n(\widehat{G})) = H_1(\widehat{G};\mathcal{K}\mathcal{H}_n(\widehat{G})).$$

Moreover, from the limit of the second row it follows that

$$\varinjlim (G_k)_H^{(n)}/(G_k)_H^{(n+1)} = \widehat{G}_H^{(n)}/\widehat{G}_H^{(n+1)}.$$

Also the limit homomorphism

$$\varinjlim \phi_{n,G_k} \colon \widehat{G}_H^{(n)}/\widehat{G}_H^{(n+1)} \longrightarrow H_1(\widehat{G};\mathcal{K}\mathcal{H}_n(\widehat{G}))$$

is equal to the homomorphism $\Phi_{n,\widehat{G}}$. So the commutative diagram in our assertion is obtained by taking the limit. This completes the proof of the assertion.

Now we prove the proposition. For the if part, note that it suffices to investigate the surjectivity of $\widehat{\sigma}_{n+1,G}$ since it is always injective. Since $\widehat{\sigma}_{n,G}$ is an isomorphism by the hypothesis, from our assertion it follows that $\widehat{\sigma}_{n+1,G}$ is surjective if and only if $\Phi_{n,\widehat{G}}$ is surjective. Since $\Phi_{n,\widehat{G}}$ is induced by the composition

$$\widehat{G}_H^{(n)} \xrightarrow{\ p\ } \widehat{G}_H^{(n)}/[\widehat{G}_H^{(n)},\widehat{G}_H^{(n)}] = H_1(\widehat{G};\mathbb{Z}\mathcal{H}_n(\widehat{G})) \xrightarrow{\ q\ } H_1(\widehat{G};\mathcal{K}\mathcal{H}_n(\widehat{G}))$$

and p is surjective, $\Phi_{n,\widehat{G}}$ is surjective if and only if q is surjective. This proves the if part.

For the only if part, note that $\widehat{\sigma}_{n+1,G}$ induces $\widehat{\sigma}_{n,G}$ on quotient groups. (To prove this, one may use the argument of the proof of our assertion above; it shows that the right square in the diagram of the assertion commutes even without assuming that $\widehat{\sigma}_{n,G}$ is an isomorphism.) Since $\widehat{\sigma}_{n+1,G}$ is surjective by the hypothesis, so is $\widehat{\sigma}_{n,G}$. Since $\widehat{\sigma}_{n,G}$ is always injective, it is an isomorphism. So from our assertion it follows that $H_1(\widehat{G};\mathbb{Z}\mathcal{H}_n(\widehat{G})) \to H_1(\widehat{G};\mathcal{K}\mathcal{H}_n(\widehat{G}))$ is surjective as in the previous paragraph. This proves the only if part. ☐

Now we investigate inductively whether $\widehat{\sigma}_{n,G}$ is an isomorphism.

Proposition 3.4. *$\widehat{\sigma}_{n,G}$ is an isomorphism between $\widehat{\mathcal{H}}_n(G)$ and $\mathcal{F}_n(G)$ for $n < 2$.*

Proof. For $n = 0$, $\widehat{\sigma}_{0,G}$ is obviously an isomorphism, being a homomorphism between trivial groups.

For $n = 1$, we consider

$$\Phi_{0,\widehat{G}} \colon H_1(\widehat{G};\mathbb{Z}) \longrightarrow H_1(\widehat{G};\mathbb{Q}).$$

By the lemma below, which is a special case of Lemma 4.10 proved later, it follows that $\Phi_{0,\widehat{G}}$ is an isomorphism:

Lemma 3.5. *For any group G, $H_1(\widehat{G};\mathbb{Z})$ is divisible.*

By Proposition 3.3, the proof of Proposition 3.4 is completed. □

However, for $n = 2$, the following result illustrates that $\widehat{\sigma}_n \colon \widehat{\mathcal{H}}_n \to \mathcal{F}_n$ is not necessarily an isomorphism:

Proposition 3.6. *For any free group F with rank > 1, $\widehat{\sigma}_{2,F} \colon \widehat{\mathcal{H}}_2(F) \to \mathcal{F}_2(F)$ is not surjective.*

From Proposition 3.6, it follows that $\widehat{\sigma}_{n,F} \colon \widehat{\mathcal{H}}_n(F) \to \mathcal{F}_n(F)$ is not surjective for any $n > 1$ (including $n = \omega$), since $\widehat{\sigma}_{n,F}$ induces a non-surjective homomorphism, namely $\widehat{\sigma}_{2,F}$, on quotient groups. Therefore we obtain the following result:

Theorem 3.7. *The container $\mathcal{F}_n$ of $\mathcal{H}_n$ is not universal for any $2 \le n \le \omega$.*

Proof of Proposition 3.6. We start with a general discussion about an arbitrary finitely presented group G. By Lemma 3.5, $H_1(\widehat{G};\mathbb{Z})$ is divisible. In fact, $H_1(\widehat{G};\mathbb{Z})$ is a $\mathbb{Q}$-module (see Lemma 4.10). So, by the definition, $\widehat{G}^{(1)}_H$ is the kernel of the surjection $\widehat{G} \to H_1(\widehat{G};\mathbb{Q}) = H_1(\widehat{G};\mathbb{Z}) \otimes \mathbb{Q} = H_1(\widehat{G};\mathbb{Z})$. It follows that $\widehat{G}^{(1)}_H$ is equal to the ordinary commutator subgroup $\widehat{G}^{(1)} = [\widehat{G},\widehat{G}]$. Also,

$$\mathcal{H}_1(\widehat{G}) = \widehat{G}/\widehat{G}^{(1)}_H = H_1(\widehat{G};\mathbb{Q}) = H_1(G;\mathbb{Q}) = \mathbb{Q}^\mu,$$

where μ is the first betti number of G. (The third equality is a well-known property of a homology localization. For concreteness we remark that it can be shown by appealing to Theorem 2.6 (2): there is a sequence $G = G_0 \to G_1 \to \cdots$ of rationally 2-connected homomorphisms with limit $\widehat{G}$.)

We consider

$$\Psi \colon H_1(\widehat{G};\mathbb{Z}H_1(\widehat{G};\mathbb{Q})) \longrightarrow H_1(\widehat{G};\mathcal{K}H_1(\widehat{G};\mathbb{Q})).$$

Since $H_1(\widehat{G};\mathbb{Q})$ is abelian, $\mathcal{K}H_1(\widehat{G};\mathbb{Q})$ is the ordinary localization $S^{-1}\cdot\mathbb{Z}H_1(\widehat{G};\mathbb{Q})$ of the commutative ring $\mathbb{Z}H_1(\widehat{G};\mathbb{Q})$ where $S = \mathbb{Z}H_1(\widehat{G};\mathbb{Q}) - \{0\}$. Moreover

$$H_1(\widehat{G};\mathcal{K}H_1(\widehat{G};\mathbb{Q})) = S^{-1}\cdot H_1(\widehat{G};\mathbb{Z}H_1(\widehat{G};\mathbb{Q}))$$

since $S^{-1}\cdot\mathbb{Z}H_1(\widehat{G};\mathbb{Q})$ is a flat $\mathbb{Z}H_1(\widehat{G};\mathbb{Q})$-module. Therefore Ψ is surjective if and only if every element in $H_1(\widehat{G};\mathbb{Z}H_1(\widehat{G};\mathbb{Q}))$ is divisible by any element in S, that is, for any $u \in H_1(\widehat{G};\mathbb{Z}H_1(\widehat{G};\mathbb{Q}))$ and $s \in S$, there exists $v \in H_1(\widehat{G};\mathbb{Z}H_1(\widehat{G};\mathbb{Q}))$ such that $s \cdot v = u$.

We will show that the divisibility criterion is not satisfied in case that G is a free group F of rank $\mu > 1$. Let x and y be two distinct generators of F. As an abuse

of notation, for an element g in F, we denote the image of g under $F \to \widehat{F}$ by g. (Indeed, it can be shown that $F \to \widehat{F}$ is injective for a free group F, although we will not use it.) Consider the element

$$u \in \widehat{F}^{(1)}/[\widehat{F}^{(1)}, \widehat{F}^{(1)}] = H_1(\widehat{F}; \mathbb{Z}H_1(\widehat{F}; \mathbb{Q}))$$

which is represented by $xyx^{-1}y^{-1} \in \widehat{F}^{(1)}$, and the element

$$s = [x] - 1 \in \mathbb{Z}H_1(F; \mathbb{Q}) = \mathbb{Z}H_1(\widehat{F}; \mathbb{Q})$$

where $[x]$ is the homology class of x. Suppose that there exists $v \in \widehat{F}^{(1)}$ such that $s \cdot v = u$ in $H_1(\widehat{F}; \mathbb{Z}H_1(\widehat{F}; \mathbb{Q}))$. Since the action of $H_1(\widehat{F}; \mathbb{Q})$ on $H_1(\widehat{F}; \mathbb{Z}H_1(\widehat{F}; \mathbb{Q}))$ is given by conjugation, we have

$$xvx^{-1}v^{-1} \equiv u = xyx^{-1}y^{-1} \mod [\widehat{F}^{(1)}, \widehat{F}^{(1)}].$$

From this it follows that $xyx^{-1}y^{-1}$ is in $\widehat{F}_3 = [\widehat{F}, [\widehat{F}, \widehat{F}]]$, the third term of the lower central series of $\widehat{F}$.

We will show that a contradiction is derived from this. Obviously, $xyx^{-1}y^{-1}$ is not in F_3, say by Hall's basis theorem. To generalize this to $\widehat{F}$, we use the rational version of Stallings' theorem: the rational derived series $G_q^{\mathbb{Q}}$ of a group G is defined inductively by

$$G_1^{\mathbb{Q}} = G, \quad G_{q+1}^{\mathbb{Q}} = \text{kernel of } G_q^{\mathbb{Q}} \longrightarrow \frac{G_q^{\mathbb{Q}}}{[G, G_q^{\mathbb{Q}}]} \longrightarrow \frac{G_q^{\mathbb{Q}}}{[G, G_q^{\mathbb{Q}}]} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}.$$

Then for any group homomorphism $\pi \to G$ which is rationally 2-connected, $\pi/\pi_q^{\mathbb{Q}} \to G/G_q^{\mathbb{Q}}$ is injective for all q [15]. We also need the following facts: obviously $G_q \subset G_q^{\mathbb{Q}}$ for any group G, and for a free group F, $F_q = F_q^{\mathbb{Q}}$ since F_q/F_{q+1} is known to be torsion free as an abelian group.

Now, applying the rational version of Stallings' theorem to our $F \to \widehat{F}$ which is rationally 2-connected, it follows that $xyx^{-1}y^{-1} \in F_3^{\mathbb{Q}} = F_3$ since $xyx^{-1}y^{-1} \in \widehat{F}_3 \subset \widehat{F}_3^{\mathbb{Q}}$. This is a contradiction. $\square$

Remark 3.8. In the proof of Proposition 3.6, we considered a particular element $s = [x] - 1$ in S to show that the divisibility criterion is not satisfied. Our argument also works for any s contained in the kernel of the augmentation homomorphism $\mathbb{Z}H_1(F; \mathbb{Q}) \to \mathbb{Z}$. Such an element s can be used for this purpose since it is invertible in the Ore localization; but it is not in the Cohn localization. As mentioned in the introduction, this fact motivates a study of a more natural series similar to the torsion-free derived series but defined using the Cohn localization instead of the Ore localization. (See also Remark 5.22 of [4].) We do not address this issue in depth in the present paper.

4. Nullhomologous equations and R-closures

Let G be a group. We call an element w in the free product $G * F\langle x_1, \ldots, x_n\rangle$ a *monomial over G in $x_1, \ldots, x_n$*, where $F\langle x_1, \ldots, x_n\rangle$ denotes the free group generated by $x_1, \ldots, x_n$. Viewing a monomial w as a word in elements of G and $x_1, \ldots, x_n$, we sometimes write $w = w(x_1, \ldots, x_n)$.

We consider systems of equations over G of the following form:

$$x_i^e = w_i(x_1, \ldots, x_n), \quad i = 1, \ldots, n$$

where $x_1, \ldots, x_n$ are considered as indeterminates, e is a nonzero integer, and $w_i(x_1, \ldots, x_n)$ is a monomial over G. An n-tuple $(g_1, \ldots, g_n)$ of elements in G is called a *solution* of the system if $x_i = g_i$ satisfies the equations, that is, g_i^e is equal to $w_i(g_1, \ldots, g_n)$ in G for all i.

Henceforth we fix a subring R of $\mathbb{Q}$. We denote by D_R the set of denominators of reduced fractional expressions of elements in R. D_R is a multiplicatively closed set.

Definition 4.1. A system $\{x_i^e = w_i(x_1, \ldots, x_n)\}$ over G is called *R-nullhomologous* if e is in D_R and each $w_i(x_1, \ldots, x_n)$ is sent to the trivial element by the canonical projection

$$G * F \longrightarrow F \longrightarrow F/[F, F] = H_1(F)$$

where $F = F\langle x_1, \ldots, x_n\rangle$.

Definition 4.2.
(1) A group A is called *R-closed* if every R-nullhomologous system over A has a unique solution in A.
(2) For a group G, an R-closed group $\widehat{G}$ equipped with a homomorphism $G \to \widehat{G}$ is called an *R-closure* of G if for any homomorphism of G into an R-closed group A, there exists a unique homomorphism $\widehat{G} \to A$ making the following diagram commute:

$$\begin{tikzcd} G \arrow[r] \arrow[d] & \widehat{G} \arrow[dl, dotted] \\ A & \end{tikzcd}$$

That is, $G \to \widehat{G}$ is the universal (initial) object in the category of homomorphisms of G into R-closed groups.

Remark 4.3. Although we do not need it in this paper, it can be seen that every R-nullhomologous system has a unique solution if and only if so does every system of the form $\{x_i^e = g_i u_i(x_1, \ldots, x_n)\}$ where $g_i \in G$, $u_i \in [F, F]$, and $e \in D_R$. This form is more similar to the equations considered in work of Farjoun–Orr–Shelah [5]. The only if part is clear. For the if part, suppose an R-nullhomologous system $\{x_i^e = w_i(x_1, \ldots, x_n)\}$ over G is given. If the variables x_i commuted with elements of G appearing in w_i, then w_i would be of the form $g_i \cdot u_i$ where $g_i \in G$ and $u_i \in [F, F]$. Therefore we can rewrite w_i as $g_i \cdot \big(\prod_j [h_{ij}, x_{ij}^{\pm 1}]\big) \cdot u_i$ where $h_{ij} \in G$. For each h_{ij},

we adjoin to the system an indeterminate y_{ij} and an equation $y_{ij} = h_{ij}$. Replacing each occurrence of h_{ij} in the original equation $x_i^e = w_i$ by the new indeterminate y_{ij}, we obtain a system of the desired form. From this the assertion follows.

The following definition generalizes the notion of invisible subgroups in Levine's work [8].

Definition 4.4. A normal subgroup N in G is called *R-invisible* if

(1) N is normally finitely generated in G, and
(2) The order of every element in $N/[G,N]$ is (finite and) in D_R.

We recall that N is said to be *normally finitely generated* in G if there exist finitely many elements $a_1, \ldots, a_n$ in G such that N is the smallest normal subgroup containing the a_i. In this case the a_i are called normal generators of N. Note that a normally finitely generated subgroup N in G is R-invisible if and only if $(N/[G,N]) \otimes_{\mathbb{Z}} R = 0$.

Remark 4.5. In his work on homology localizations, Bousfield called a normal subgroup N in G *π-perfect* if $N = [G,N]$ [1]. For our purpose, we need to modify it regarding the coefficient R and the finiteness assumption as in the above definition. When $R = \mathbb{Z}$, our definition agrees with the definition of an *invisible* subgroup due to Levine [8].

In what follows we discuss some useful relationships between R-invisible subgroups and R-nullhomologous systems.

Lemma 4.6. *Suppose $\phi\colon G \to A$ is a homomorphism into an R-closed group A. Then every R-invisible subgroup N in G is contained in the kernel of ϕ.*

Proof. Choose normal generators $a_1, \ldots, a_n$ of N. Since the order of a_i is in D_R, there exist an element $e \in D_R$ such that $a_i^e \in [G,N]$ for all i. So we can write a_i^e as a product of commutators $[b_{ij}, c_{ij}]$ where $b_{ij} \in N$, $c_{ij} \in G$. Furthermore b_{ij} can be written as a product of conjugates of the a_k. Replacing each occurrence of a_k in this expression of b_{ij} by an indeterminate x_k and plugging the result into the above expression of a_i^e, we obtain a word $w_i(x_1, \ldots, x_n)$ in $G * F$, F is the free group generated by the x_i, such that the system

$$x_i^e = w_i(x_1, \ldots, x_n), \qquad i = 1, \ldots, n$$

has two sets of solutions, $\{x_i = 1\}$ and $\{x_i = a_i\}$. It is easily seen that this system is R-nullhomologous.

Denote by w_i^ϕ the image of w_i under $G * F \to A * F$. It gives rise to an R-nullhomologous system

$$x_i^e = w_i^\phi(x_1, \ldots, x_n), \qquad i = 1, \ldots, n$$

over A, which has two solution sets $\{x_i = 1\}$ and $\{x_i = \phi(a_i)\}$. By the uniqueness of a solution over A, it follows that $\phi(a_i) = 1$. Thus $\phi(N)$ is trivial. $\square$

Lemma 4.7. *If N_1 and N_2 are R-invisible subgroups in G, then $N = N_1 N_2$ is also R-invisible in G.*

Proof. Obviously N is normally finitely generated. For any $n \in N$, write $n = n_1 n_2$ where $n_i \in N_i$. Since N_i is R-invisible, there exist $e \in D_R$ such that $n_i^e \in [G, N_i]$. Then

$$(n_1 n_2)^e \equiv n_1^e n_2^e \equiv 1 \mod [G, N].$$

This shows that the order of $n[G, N]$ in $N/[G, N]$ is (a divisor of) e. □

Lemma 4.8. *Suppose G is a group and N is the union of all R-invisible subgroups in G. Then G/N has no nontrivial R-invisible subgroup.*

Proof. First of all, N is a normal subgroup by the previous lemma. Suppose H/N is R-invisible in G/N for some $H \subset G$. Choose a finite normal generator set $\{h_i N\}$ of H/N, $h_i \in H$. It suffices to show that $h_i \in N$ for each i. For some $e \in D_R$, $h_i^e \in [H/N, G/N]$. Therefore $h_i^e \in n_i[H, G]$ for some $n_i \in N$. By the previous lemma, there exists an R-invisible subgroup K in G such that $n_i \in K$ for all i. Then the normal subgroup K_1 generated by K and the h_i is R-invisible in G. It follows that $h_i \in K_1 \subset N$. □

Lemma 4.9. *If G has no nontrivial R-invisible subgroup, then any R-nullhomologous system over G has at most one solution.*

Proof. Suppose $S = \{x_i^e = w_i(x_1, \ldots, x_n)\}$ is R-nullhomologous and $\{x_i = a_i\}$ and $\{x_i = b_i\}$ are solutions of S over G. Let N be the normal subgroup in G generated by the $a_i b_i^{-1}$, and let

$$u_i(x_1, \ldots, x_n) = w_i(x_1 b_1, \ldots, x_n b_n) b_i^{-e}.$$

Since $u_i(1, \ldots, 1) = w_i(b_1, \ldots, b_n) b_i^{-e} = 1$, we can write u_i as a word of the form

$$\begin{aligned} u_i &= \prod_j g_{ij} x_{ij}^{\pm 1} g_{ij}^{-1} \\ &= \Big(\prod_j (x_{i1}^{\pm 1} \cdots x_{i,j-1}^{\pm 1}) [g_{ij}, x_{ij}^{\pm 1}] (x_{i1}^{\pm 1} \cdots x_{i,j-1}^{\pm 1})^{-1} \Big) \prod_j x_{ij}^{\pm 1} \end{aligned}$$

where $g_{ij} \in G$ and $x_{ij} = x_{k_{ij}}$ for some k_{ij}. Furthermore, each u_i is killed by $G * F \to F/[F, F]$ where $F = F\langle x_1, \ldots, x_n \rangle$, since so is w_i. It follows that $\prod_j x_{ij}^{\pm 1}$, the last term of the above expression, is contained in $[F, F]$. Therefore

$$\begin{aligned} a_i^e b_i^{-e} &= w_i(a_1, \ldots, a_n) b_i^{-e} \\ &= u_i(a_1 b_1^{-1}, \ldots, a_n b_n^{-1}) \in [G, N]. \end{aligned}$$

Now we have

$$\begin{aligned} (a_i b_i^{-1})^e &\equiv (a_i b_i^{-1})^e \cdot a_i^{-e} b_i^e \\ &\equiv (a_i b_i^{-1})^{e-1} \cdot a_i b_i^{-1} \cdot a_i^{-e} b_i^e \\ &\equiv (a_i b_i^{-1})^{e-1} \cdot a_i^{-e} \cdot a_i b_i^{-1} \cdot b_i^e \\ &\equiv (a_i b_i^{-1})^{e-1} \cdot a_i^{-e+1} b_i^{e-1} \equiv \cdots \equiv 1 \mod [G, N] \end{aligned}$$

This shows that N is R-invisible. By the hypothesis, N is trivial and $a_i = b_i$. □

As an immediate consequence of the definition, we have the following divisibility result:

Lemma 4.10. *If A is R-closed, then $H_1(A;\mathbb{Z})$ is an R-module.*

Proof. Let g be an element in A, and let e be an element in D_R. Consider the equation $x^e = g$. Since it is R-nullhomologous, there is a solution $x = h$ in A. It follows that the homology class of g is divisible by e. □

5. R-closures and localizations

We begin this section by recalling the definition of a localization. In general, we think of a category $\mathcal{C}$ and a class of morphisms Ω in $\mathcal{C}$.

Definition 5.1. (1) An object A in $\mathcal{C}$ is called *local* with respect to Ω if for any morphism $\pi \to G$ in Ω and any morphism $\pi \to A$, there exists a unique morphism $G \to A$ making

$$\begin{tikzcd} \pi \arrow[r] \arrow[d] & G \arrow[dl, dotted] \\ A & \end{tikzcd}$$

commute.

(2) A *localization* with respect to Ω is a pair (E, p) of a functor $E\colon \mathcal{C} \to \mathcal{C}$ and a natural transformation $p\colon \mathrm{id}_{\mathcal{C}} \to E$ (that is, each object G is equipped with a morphism $p_G\colon G \to E(G)$) such that for any morphism $G \to A$ into a local object A, there is a unique morphism $E(G) \to A$ making

$$\begin{tikzcd} G \arrow[r] \arrow[d] & E(G) \arrow[dl, dotted] \\ A & \end{tikzcd}$$

commute.

Of course our main interest is the localization of groups with respect to the class Ω^R; recall that Ω^R is the class of group homomorphisms $\phi\colon \pi \to G$ where π is finitely generated, G is finitely presented, and ϕ is 2-connected on R-homology. From now on local groups and localizations are always with respect to our Ω^R.

Theorem 5.2. *A group A is R-closed if and only if A is local with respect to Ω^R.*

Proof. Suppose that A is R-closed. To show that A is local, suppose that a morphism $\alpha\colon \pi \to G$ in Ω^R and a morphism $\phi\colon \pi \to A$ are given. We will show that there exists a unique morphism $\varphi\colon G \to A$ that $\phi\colon \pi \to A$ factors through.

Choose generators $h_1, \ldots, h_n$ of G. Since α induces an isomorphism

$$H_1(\pi; R) = \frac{\pi}{[\pi,\pi]} \underset{\mathbb{Z}}{\otimes} R \longrightarrow \frac{G}{[G,G]} \underset{\mathbb{Z}}{\otimes} R,$$

it follows that for any element x in $G/[G,G]$, x^r is contained in the image of $\pi/[\pi,\pi]$ for some $r \in D_R$. Therefore there exists $e \in D_R$ such that each h_i^e can be written as

$$h_i^e = \alpha(g_i) \cdot \prod_j [u_{ij}, v_{ij}]$$

where $g_i \in \pi$ and $u_{ij} = u_{ij}(h_1, \ldots, h_n)$, $v_{ij} = v_{ij}(h_1, \ldots, h_n)$ are words in $h_1, \ldots, h_n$.

Consider the system S of equations $x_i^e = w_i(x_1, \ldots, x_n)$ where the element w_i in $G * F\langle x_1, \ldots, x_n\rangle$ is given by

$$w_i(x_1, \ldots, x_n) = g_i \cdot \prod_j [u_{ij}(x_1, \ldots, x_n), v_{ij}(x_1, \ldots, x_n)].$$

Then S is R-nullhomologous.

We associated to the system S a new group π_S obtained by "adding" to π a solution $\{z_i\}$ to S; formally, it is defined to be a amalgamated product of π and $F\langle z_1, \ldots, z_n\rangle$:

$$\pi_S = \langle \pi, z_1, \ldots, z_n \mid z_i^e = w_i(z_1, \ldots, z_n),\ i = 1, \ldots, n\rangle.$$

Note that, since $e \in D_R$, the canonical homomorphism $\pi \to \pi_S$ is 2-connected on R-homology; it can be seen by computing $H_1(\pi_S; R)$ and $H_2(\pi_S; R)$ using the complex obtained by attaching to $K(\pi, 1)$ 1-cells and 2-cells corresponding the new generators and relations.

$\pi \to G$ and $z_i \to h_i \in G$ induce a surjection $\beta\colon \pi_S \to G$. Since A is R-closed, there is a unique solution $\{x_i = a_i\}$ of the R-nullhomologous system $S^\phi = \{x_i^e = w_i^\phi(x_1, \ldots, x_n)\}$ over A, which is the image of S under ϕ. $\phi\colon \pi \to A$ and $z_i \to a_i \in A$ induce a homomorphism $\phi_S\colon \pi_S \to A$.

$$\begin{tikzcd}
\pi \arrow[r] \arrow[d, "\phi"'] & \pi_S \arrow[r, "\beta"] \arrow[dl, "\phi_S"] & G \arrow[dll, dotted, "\varphi"] \\
A & &
\end{tikzcd}$$

We will show that ϕ_S factors through a homomorphism $\varphi\colon G \to A$. Let N be the kernel of β. Applying the Stallings exact sequence [15] to β which is surjective, we obtain a long exact sequence

$$H_2(\pi_S; R) \longrightarrow H_2(G; R) \longrightarrow \frac{N}{[\pi_S, N]} \otimes_{\mathbb{Z}} R \longrightarrow H_1(\pi_S; R) \longrightarrow H_1(G; R).$$

Since $\alpha\colon \pi \to G$ and $\pi \to \pi_S$ are 2-connected on R-homology, so is β. Thus $(N/[\pi_S, N]) \otimes R = 0$. Since π_S is finitely generated and G is finitely presented, N is finitely normally generated in π_S. This shows that N is R-invisible in π_S. By

Lemma 4.6, $\phi_S(N)$ is trivial. It follows that ϕ_S induces a homomorphism $\varphi\colon G \to A$ as desired.

If another homomorphism $\varphi'\colon G \to A$ satisfies $\phi = \varphi' \circ \alpha$, then $\{x_i = \varphi'(h_i)\}$ is a solution of the system S^ϕ. By the uniqueness of a solution, we have $\varphi(h_i) = a_i = \varphi'(h_i)$, that is, $\varphi = \varphi'$. This completes the proof of the only if part.

For the converse, suppose that A is local, and an R-nullhomologous system $S = \{x_i^e = w_i(x_1, \ldots, x_n)\}$ over A is given. There are finitely many elements in A which appears in the words w_i. Let G be the free group generated by (symbols associated to) these elements, and $\phi\colon G \to A$ be the natural homomorphism. Lifting S, we obtain a system $S' = \{x_i^e = w_i(x_1, \ldots, x_n)\}$ over G which is sent to S by ϕ. Consider the group $G_{S'}$ obtained by "adding a solution $\{z_i\}$ to the system S'" as before. Then $G_{S'}$ is finitely presented and the canonical homomorphism $\alpha\colon G \to G_{S'}$ is 2-connected. Since A is local, there exists a unique homomorphism $\varphi\colon G_{S'} \to A$ making

$$\begin{tikzcd} G \arrow[r, "\alpha"] \arrow[d, "\phi"'] & G_{S'} \arrow[dl, dotted, "\varphi"] \\ A & \end{tikzcd}$$

commute. Now $\{\varphi(z_i)\}$ is a solution of S over A.

If there is another solution $\{x_i = a_i\}$ of S, then $x_i \to a_i$ gives rise to another homomorphism $\varphi'\colon G_{S'} \to A$ making the above diagram commute. By the uniqueness of φ, $\varphi' = \varphi$, and therefore, $a_i = \varphi(z_i)$. This completes the proof. □

6. Existence of R-closures

Theorem 6.1. *For any subring R of $\mathbb{Q}$ and any group G, there is an R-closure $G \to \widehat{G}$.*

Proof. Basically the construction consists of two parts: adjoin solutions repeatedly so that every system has at least one solution eventually, and take a quotient of the resulting group to identify different solutions if any.

This idea is formalized as follows. We construct a sequence $G_0, G_1, \ldots$ of groups inductively. Let $G_0 = G$. Suppose G_k has been defined. Let $\mathcal{S}_k$ be the set of all R-nullhomologous systems over G_k. We associate a symbol z_i to an indeterminate x_i of a system in $\mathcal{S}_k$, and let F_k be the free group generated by all the symbols z_i. Let $G_{k+1} = G_k * F_k$ modulo the relations corresponding the systems in $\mathcal{S}_k$, that is, each equation $x_i^e = w_i(x_1, \ldots, x_n)$ gives rise to a defining relation $z_i^e = w_i(z_1, \ldots, z_n)$ of G_{k+1}. Let $\bar{G} = \varinjlim G_k$. Let N be the union of all R-invisible subgroups in $\bar{G}$, and finally let $\widehat{G} = \bar{G}/N$.

We will show that the canonical homomorphism $\Phi\colon G \to \widehat{G}$ is an R-closure of G. First we claim that $\widehat{G}$ is R-closed. For the existence of a solution, suppose that $S = \{x_i^e = w_i(x_1, \ldots, x_n)\}$ be an R-nullhomologous system over $\widehat{G}$. Since the w_i involve only finitely many elements of $\widehat{G}$, S lifts to an R-nullhomologous system

over some G_k, that is, each w_i is the image of an element w_i' in $G_k * F$ where $F = F\langle x_1, \ldots, x_n\rangle$. Sending w_i' via $G_k * F \to G_{k+1} * F$, we obtain an R-nullhomologous system over G_{k+1} which has a solution $\{x_i = a_i\}$ by our construction of G_{k+1}; recall that G_{k+1} is obtained by adjoining solutions of all R-nullhomologous systems over G_k. Obviously the image of the a_i in $\widehat{G}$ is a solution of the given system S. On the other hand, $\widehat{G}$ has no nontrivial R-invisible subgroup by Lemma 4.8. From Lemma 4.9, the uniqueness of a solution follows. This proves the claim.

Now it remains to show that $\Phi\colon G \to \widehat{G}$ is a universal (initial) object. Suppose that $\phi\colon G \to A$ is a homomorphism of G into an R-closed group A. Since G_1 is obtained from $G = G_0$ by adjoining solutions of R-nullhomologous systems, there exists a unique homomorphism $\phi_1\colon G_1 \to A$ making the below diagram commute; ϕ_1 is defined by sending new generators z_i of G_1 associated to a system $S = \{x_i = w_i(x_1, \ldots, x_n)\}$ over G_0 to the solution of S^ϕ over A, and the uniqueness of ϕ_1 follows from the uniqueness of a solution over A.

$$\begin{tikzcd}
G \arrow[r] \arrow[d, "\phi"'] & G_1 \arrow[r] \arrow[dl, dotted, "\phi_1"] & G_2 \arrow[r] \arrow[dll, dotted, "\phi_2"] & \cdots \arrow[r] & \bar{G} = \varinjlim G_k \\
A
\end{tikzcd}$$

Repeating the same argument, we can inductively construct a sequence of homomorphisms $\phi_k\colon G_k \to A$ which make the diagram commute. Passing to the limit, the ϕ_k induce a homomorphism $\bar\phi\colon \bar{G} \to A$. By Lemma 4.6, φ' kills each R-invisible subgroup, and so N is contained in the kernel of $\bar\phi$. It follows that $\bar\phi$ gives rise to a homomorphism $\varphi\colon \widehat{G} = \bar{G}/N \to A$ such that $\varphi \circ \Phi = \phi$.

Suppose another homomorphism $\varphi'\colon \widehat{G} \to A$ satisfies $\varphi' \circ \Phi = \phi$. Consider the composition $\phi_k'\colon G_k \to \widehat{G} \xrightarrow{\varphi'} A$. Then the ϕ_k make the above diagram commute as well as the ϕ_k. From the uniqueness of a solution over A, it follows that $\phi_k = \phi_k'$ for every k. Passing to the limit, it follows that $\varphi = \varphi'$. This completes the proof. $\square$

Remark 6.2. There is an alternative construction of an R-closure: let $G_0 = G$ as before, and assuming G_k has been defined, consider the group obtained from G_k by adjoining solutions of all R-nullhomologous systems, and let G_{k+1} be its quotient by the union of all R-invisible subgroups. Then it can be proved that $\varinjlim G_k$ is an R-closure of G.

Corollary 6.3. *There is a localization (E, p) with respect to Ω^R.*

Proof. For each group G, define $E(G) = \widehat{G}$ and $p_G\colon G \to \widehat{G}$ to be the homomorphism Φ constructed above. For any homomorphism $\phi\colon \pi \to G$, the composition $\pi \to G \to E(G) = \widehat{G}$ is a homomorphism of π into an R-closed group. By the universal property of the R-closure $\pi \to E(\pi) = \widehat{\pi}$, there is a unique homomorphism $E(\pi) \to E(G)$ that the composition factors through; we denote this homomorphism by $E(\phi)\colon E(\pi) \to E(G)$. One can check that E is a functor and p is a natural transformation by using the universal property of R-closures in a straightforward way.

Finally, since $G \to E(G)$ is an R-closure, it is initial among homomorphisms of G into local groups, by Theorem 5.2. $\square$

We sometimes denote the induced homomorphism $E(\phi)$ by $\widehat{\phi}$.

From the universal property of the R-closure, a natural isomorphism theorem follows:

Proposition 6.4. *If $\alpha\colon \pi \to G$ is in Ω^R, then the induced homomorphism $\widehat{\alpha}\colon \widehat{\pi} \to \widehat{G}$ is an isomorphism.*

Proof. We consider the following diagram:

$$\begin{tikzcd} \pi \arrow[r, "\alpha"] \arrow[d, "p_\pi"'] & G \arrow[d, "p_G"] \arrow[dl, dotted, "\phi"'] \\ \widehat{\pi} \arrow[r, "\widehat{\alpha}"'] & \widehat{G} \end{tikzcd}$$

Since α is in Ω^R and $\widehat{\pi}$ is local, there is $\phi\colon G \to \widehat{\pi}$ such that $\phi\alpha = p_\pi$. Since $\widehat{\pi}$ is local and $\widehat{G}$ is the localization, there is $\psi\colon \widehat{G} \to \widehat{\pi}$ such that $\psi p_G = \phi$. We will show that ψ is an inverse of $\widehat{\alpha}$. Observe that $(\widehat{\alpha}\psi p_G)\alpha = p_G\alpha$. Since $\widehat{G}$ is local and $\alpha \in \Omega^R$, we have $\widehat{\alpha}\psi p_G = p_G$. It follows that $\widehat{\alpha}\psi = \mathrm{id}_{\widehat{G}}$ by the uniqueness of a map $\beta\colon \widehat{G} \to \widehat{G}$ such that $\beta p_G = p_G$. On the other hand, since $\psi\widehat{\alpha}p_\pi = \psi p_G\alpha = \phi\alpha = p_\pi$, $\psi\widehat{\alpha} = \mathrm{id}_{\widehat{\pi}}$ by a similar uniqueness argument. $\square$

Remark 6.5.
(1) For subrings $R \subset S \subset \mathbb{Q}$, temporarily denote by $\widehat{G}^R$ and $\widehat{G}^S$ the R- and S-closures of a group G, respectively. Then, since $\Omega^R \subset \Omega^S$, there is a natural transformation $\widehat{G}^R \to \widehat{G}^S$ making the following diagram commute:

$$\begin{tikzcd} G \arrow[rr] \arrow[dr] & & \widehat{G}^S \\ & \widehat{G}^R \arrow[ur] & \end{tikzcd}$$

(2) Similar conclusion holds for Levine's algebraic closure defined in [8], which is a localization with respect to the collection Ω^{Levine} of group homomorphisms $\alpha\colon \pi \to G$ such that π, G are finitely presented, α is (integrally) 2-connected, and G is normally generated by the image of α. Namely, denoting Levine's algebraic closure by $\widehat{G}^{Levine}$, there is a natural transformation $\widehat{G}^{Levine} \to \widehat{G}^{\mathbb{Z}}$ making a similar diagram commute, since $\Omega^{Levine} \subset \Omega^{\mathbb{Z}}$.

We finish this section with some results on the R-closure of a finitely presented group.

Proposition 6.6. *If G is finitely presented, then there is a sequence*

$$G = P_0 \longrightarrow P_1 \longrightarrow P_2 \longrightarrow \cdots$$

of homomorphisms in Ω^R (in particular, each P_k is finitely presented) such that the limit homomorphism $G = P_0 \to \varinjlim P_k$ is an R-closure, that is, $\widehat{G} \cong \varinjlim P_k$.

Proof. We use the notations of the proof of Theorem 6.1. Recall that $\mathcal{S}_k$ is the set of all R-nullhomologous systems over G_k. We denote $\mathcal{S} = \bigcup \mathcal{S}_k$. Then an element $S = \{x_i = w_i\}$ in $\mathcal{S}$ is a system over G_p for some p. By our construction of G_p, each w_i can be viewed as a word in indeterminates $x_1, x_2, \ldots$ and solutions of other systems over some G_q where $q < p$.

Since $G_0 = G$ is countable, an induction shows that G_k and $\mathcal{S}_k$ are always countable. Thus the union $\mathcal{S}$ of all $\mathcal{S}_k$ is countable. From this it can be seen that we can enumerate elements of $\mathcal{S}$ as a sequence $T_1, T_2, \ldots$ of systems which satisfies the following: suppose the system $T_k = \{x_i = w_i\}$ is over G_p. Then each w_i involves only elements in G_p that can be expressed as a product of solutions (and their inverses) of other systems T_q such that $q < k$. In other words, the system T_k can be lifted to a system over the group

$$Q_{k-1} = (\cdots((G_{T_1})_{T_2})\cdots)_{T_{k-1}}.$$

(Recall our notation that Q_{k-1} is the group obtained from G by adjoining the solutions of the systems $T_1, T_2, \ldots, T_k$.) Note that there is a canonical map $Q_k \to Q_{k+1}$; it is in Ω^R as we discussed before. Furthermore, it is obvious that $\varinjlim Q_k \cong \bar{G} = \varinjlim G_k$.

We claim that for each k there is an R-invisible subgroup N_k in Q_k such that

(1) $Q_k \to Q_{k+1}$ sends N_k into N_{k+1}, and
(2) for any R-invisible subgroup K in $\bar{G}$, there is k such that K is contained in the normal subgroup of G generated by the image of N_k under $Q_k \to \bar{G}$.

For, since $\bar{G}$ is countable, we can arrange all R-invisible subgroups in $\bar{G}$ as a sequence, and by appealing to Lemma 4.7, we can produce an increasing sequence $L_1 \subset L_2 \subset \cdots$ of R-invisible subgroups in $\bar{G}$ such that every R-invisible subgroup in $\bar{G}$ is contained in some L_k. Since each L_k R-invisible, it is normally generated by finitely many elements b_i which satisfy equations of the following form:

$$b_i^e = \prod_j h_j [b_{i_j}, g_j] h_j^{-1} \quad \text{where } h_j, g_j \in \bar{G},\ e \in D_R.$$

All the $b_i, h_j, g_j \in \bar{G}$ which appear in this expression can be lifted to Q_n for some n in such a way that the equations can also be lifted, that is, replacing these elements in the equation by the lifts, we again obtain an equality in Q_n. Choosing a subsequence of $\{Q_k\}$, we may assume that $n = k$. Then the lifts of b_i in Q_k generate an R-invisible subgroup N'_k in Q_k. Let N_k be the union of the image of N'_ℓ under $Q_\ell \to Q_k$ where ℓ runs over $1, \ldots, k$. Now the claim follows.

Let $P_k = Q_k/N_k$. By our claim (1), $Q_k \to Q_{k+1}$ induces $P_k \to P_{k+1}$. We will show that these groups P_k has the desired properties. First, $P_k \to P_{k+1}$ is in Ω^R since $Q_k \to Q_{k+1}$ is 2-connected and so is $Q_k \to P_k$ by the Stallings exact sequence.

So it remains to show that $\varinjlim P_k \cong \widehat{G}$. For each k the composition

$$j_k\colon Q_k \longrightarrow \bar{G} \longrightarrow \widehat{G}$$

gives rise to $P_k \to \widehat{G}$ since it kills the R-invisible subgroup N_k. These morphisms induces $\varinjlim P_k \to \widehat{G}$, which is obviously surjective. To show that it is injective, suppose that j_k sends an element $x \in Q_k$ into an R-invisible subgroup K in $\bar{G}$. We may assume that K is contained in the normal closure of $j_k(N_k)$ by our claim (2) above. Then $j_k(x)$ is of the form $\prod_j g_j y_j g_j^{-1}$, $g_j \in \bar{G}$, $y_j \in j_k(N_k)$. By choosing a sufficiently large k, we may assume that every g_j is in the image of j_k. So $j_k(x) \in j_k(N_k)$. By choosing a larger k, we may assume that $x \in N_k$. Now it follows that x is sent to the identity in P_k. It proves that $\varinjlim P_k \to \widehat{G}$ is injective. $\square$

Proposition 6.7. *Suppose that $\pi_0 \to \pi_1 \to \pi_2 \to \cdots$ and $G_0 \to G_1 \to G_2 \to \cdots$ are sequences of homomorphisms in Ω^R such that $\pi_0 \to \varinjlim \pi_k$ and $G_0 \to \varinjlim G_k$ are the R-closures. Then for any $\phi\colon \pi_0 \to G_0$, there exist a sequence $G_0 = P_0 \to P_1 \to P_2 \to \cdots$ of homomorphisms in Ω^R which fits into the commutative diagram*

$$\begin{tikzcd}
\pi_0 \arrow[r] \arrow[d, "\phi"'] & \pi_1 \arrow[r] \arrow[d] & \cdots \arrow[r] & \pi_k \arrow[r] \arrow[d] & \pi_{k+1} \arrow[r] \arrow[d] & \cdots \arrow[r] & \varinjlim \pi_k = \widehat{\pi} \arrow[d] \\
P_0 \arrow[r] \arrow[d, equal] & P_1 \arrow[r] & \cdots \arrow[r] & P_k \arrow[r] & P_{k+1} \arrow[r] & \cdots \arrow[r] & \varinjlim P_k \\
G_0 \arrow[r] & G_1 \arrow[r] \arrow[u] & \cdots \arrow[r] & G_k \arrow[r] \arrow[u] & G_{k+1} \arrow[r] \arrow[u] & \cdots \arrow[r] & \varinjlim G_k = \widehat{G} \arrow[u]
\end{tikzcd}$$

in such a way that each $G_k \to P_k$ is in Ω^R and $\widehat{G} \to \varinjlim P_k$ is an isomorphism, that is, $G_0 = P_0 \to \varinjlim P_k$ is an R-closure.

Proof. We will construct inductively a sequence $n_0 < n_1 < \cdots$ such that $P_k = G_{n_k}$ has the desired property. Let $n_0 = 0$. As our induction hypothesis, suppose that $n_0, \ldots, n_k$ have been chosen in such a way that the following diagram commutes, where $\widehat{\phi}$ is the map induced by ϕ. (Note that $n_k \geq k$ automatically.)

$$\begin{tikzcd}
\pi_0 \arrow[r] \arrow[d, "\phi"'] & \cdots \arrow[r] & \pi_k \arrow[r] \arrow[d] & \pi_{k+1} \arrow[r] \arrow[ddrrr, dotted] & \cdots \arrow[rrr] & & & \varinjlim \pi_k \arrow[dd, "\widehat{\phi}"] \\
G_{n_0} \arrow[r] \arrow[d, equal] & \cdots \arrow[r] & G_{n_k} \arrow[drrr, equal] & & & & & \\
G_0 \arrow[r] & \cdots \arrow[r] & G_k \arrow[r] \arrow[u] & G_{k+1} \arrow[r] & \cdots \arrow[r] & G_{n_k} \arrow[r] & \cdots \arrow[r] & \varinjlim G_k
\end{tikzcd}$$

Since π_{k+1} is finitely generated, the composition $\pi_{k+1} \to \varinjlim \pi_k \to \varinjlim G_k$ factors through G_r for some $r > n_k$. Note that the two compositions $\pi_k \to \pi_{k+1} \to G_r$

and $\pi_k \to G_{n_k} \to G_r$ may not be identical. However, composing $G_r \to \varinjlim G_k$ with them, we obtain the same maps. Since π_k is finitely generated, it follows that $\pi_k \to \pi_{k+1} \to G_s$ and $\pi_k \to G_{n_k} \to G_s$ are identical for some $s \geq r$. We choose s as n_{k+1}. Our induction hypothesis is maintained so that the construction of $\{n_k\}$ can be continued.

Now, letting $P_k = G_{n_k}$, obviously $P_k \to P_{k+1}$, $G_k \to P_k$ are in Ω^R, and $\varinjlim P_k \cong \varinjlim G_k$. $\square$

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